

Quantum isotonic nonlinear oscillator as a Hermitian counterpart of Swanson Hamiltonian and pseudo-supersymmetry

Özlem Yeşiltaş¹

Department of Physics, Faculty of Science, Gazi University, 06500 Ankara, Turkey

Abstract

Within the ideas of pseudo-supersymmetry, we have studied a non-Hermitian Hamiltonian $H_- = \omega(\xi^\dagger \xi + \frac{1}{2}) + \alpha \xi^2 + \beta \xi^{\dagger 2}$, where $\alpha \neq \beta$ and ξ is a first order differential operator, to obtain the partner potentials $V_+(x)$ and $V_-(x)$ which are new isotonic and isotonic nonlinear oscillators, respectively, as the Hermitian equivalents of the non-Hermitian partner Hamiltonians H_\pm . We have provided an algebraic way to obtain the spectrum and wavefunctions of a nonlinear isotonic oscillator. The solutions of $V_-(x)$ which are Hermitian counterparts of Swanson Hamiltonian are obtained under some parameter restrictions that are found. Also, we have checked that if the intertwining operator satisfies $\eta_1 H_- = H_+ \eta_1$, where $\eta_1 = \rho^{-1} \mathcal{A} \rho$ and \mathcal{A} is the first order differential operator, which factorizes Hermitian equivalents of H_\pm .

¹e-mail : yesiltas@gazi.edu.tr

1 Introduction

A class of quantum potentials known as conditionally exactly solvable (CES) models which include one or more specific fixed coupling constants was introduced by Dutra [1]. Since then, CES models have attracted considerable attention [2, 3, 4, 5, 6, 7, 8, 9]. Recently, a non-polynomial, one dimensional isotonic nonlinear oscillator potential [6, 7, 8, 9] and its generalization to a d dimensional model have been studied [10]. On the other hand, specific symmetries of the isotonic potential have been investigated [11]. Previously, a CES model which is an isotonic nonlinear oscillator potential constructed by a superpotential in [3]. The isotonic potential which is a harmonic oscillator with a barrier potential is given by [4, 5]

$$V(x) = \frac{1}{2}x^2 + \frac{1}{2}gx^{-2} \quad (1)$$

where $g > -\frac{1}{2}$. Classically, isotonic potential model has a property which is the family of periodic solutions with the same period [12] and quantum mechanically the energy spectrum is equidistant [13]. Recently, for the Schrödinger equation, Cariñena et al investigated the solutions of

$$U(x) = x^2 + 8\frac{2x^2 - 1}{(2x^2 + 1)^2} \quad (2)$$

where the new additional term is non-polynomial and it is the sum of two centripetal barriers in the complex plane [6]. Later Fellows and Smith showed that this potential is a supersymmetric partner of a harmonic oscillator [7]. Within position dependent mass approach, it is shown that the source and the target potentials share the same energy where the isotonic potential is the reference potential [9]. At the same time, the fundamental framework of pseudo-Hermitian quantum mechanics has been proposed in [14] and the idea of non-Hermitian Hamiltonians has spread to many branches of physics such as electrodynamics [15], biological physics [16], quantum information [17] and theoretical physics [18, 19, 20, 21].

In this study, we discuss the isotonic oscillator with a non-polynomial term which is a Hermitian counterpart of non-Hermitian Swanson Hamiltonian [22, 23, 24, 25, 26, 27, 28, 29], in other words, our model can be a generalization of the non-Hermitian models for a nonlinear isotonic potential within the framework of pseudo-supersymmetry [30, 31, 32]. We shall also seek for the solutions of the corresponding system. The plan of the paper is as follows. In section 2, we introduce a brief review for factorization and hierarchy of Hamiltonians. Section 3 devoted to non-Hermitian partner Hamiltonians H_{\pm} which are isospectral except for the ground state. Pseudo- supersymmetry relations between the partner Hamiltonians are discussed. In section 4, the solutions of the Hermitian equivalent Hamiltonians of H_{\pm} are introduced.

The aim of this work devotes us to use factorization approach. As is well known, the superalgebra constructed by the super Hamiltonian \mathcal{H}_s and the supercharges Q, Q^{\dagger} is [33]

$$\mathcal{H}_s = \{Q, Q^{\dagger}\} \quad (3)$$

where Q, Q^{\dagger} commute with \mathcal{H}_s and ' \dagger ' is the Hermitian conjugation:

$$Q^2 = Q^{\dagger 2} = [Q, \mathcal{H}_s] = [Q^{\dagger}, \mathcal{H}_s] = 0. \quad (4)$$

The supercharges are given by the following matrices,

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad Q^{\dagger} = \begin{pmatrix} 0 & A^{\dagger} \\ 0 & 0 \end{pmatrix}. \quad (5)$$

Here, \mathcal{H}_s is diagonal that can be factorized in the manner

$$\mathcal{H}_s = \begin{pmatrix} \mathcal{H}_+ & 0 \\ 0 & \mathcal{H}_- \end{pmatrix}, \quad \mathcal{H}_- = A^\dagger A, \quad \mathcal{H}_+ = AA^\dagger \quad (6)$$

where A and A^\dagger are first order differential operators given by a more general realization [34]

$$A = a(x)\frac{d}{dx} + b(x), \quad A^\dagger = -a(x)\frac{d}{dx} + b(x) - a'(x), \quad ' = \frac{d}{dx} \quad (7)$$

where $b(x)$ is the superpotential related to the partner potentials

$$U_+(x) = U_-(x) + 2b'(x)a(x) - a(x)a''(x), \quad (8)$$

which are elements of the partner Hamiltonians:

$$\mathcal{H}_\pm = -\frac{d}{dx}a^2(x)\frac{d}{dx} + U_\pm(x). \quad (9)$$

The ground states can be introduced as

$$A\psi_0^- = 0 \quad \text{or} \quad A^\dagger\psi_0^+ = 0 \quad (10)$$

where ψ_n^\pm ($n = 0, 1, \dots$) stands for the bound state wavefunctions for \mathcal{H}_\pm . Let E_n^\pm are the energy eigenvalues of \mathcal{H}_\pm , then we suppose that the \mathcal{H}_- is known and has normalizable vanishing ground state energy in the unbroken SUSY case. In this manner, we have

$$\psi_n^-(x) = (E_n^+)^{-\frac{1}{2}} A^\dagger \psi_n^+, \quad \psi_n^+ = (E_n^-)^{-\frac{1}{2}} A \psi_n^-, \quad E_n^+ = E_{n+1}^-, \quad E_0^- = 0. \quad (11)$$

2 Non-Hermitian partner Hamiltonians

In order to describe a physical system, a non-Hermitian Hamiltonian H_- should be η -pseudo Hermitian [14],

$$H_-^\dagger = \eta H_- \eta^{-1} \quad (12)$$

where η is a linear, invertible operator. If η is not a Hermitian operator, then the Hamiltonian is known as weakly pseudo-Hermitian [35]. To find a Hermitian Hamiltonian h_- , one can use a similarity transformation $h_- = \rho H_- \rho^{-1}$ and the metric operator is obtained using $\eta = \rho^2$. Consider the ladder operators ξ, ξ^\dagger that are first order differential operators

$$\xi = a(x)\frac{d}{dx} + b(x), \quad \xi^\dagger = -a(x)\frac{d}{dx} + b(x) - a'(x) \quad (13)$$

which will be used in a pseudo-Hermitian quadratic Hamiltonian which is known as Swanson model [22]:

$$H_- = \omega(\xi^\dagger \xi + \frac{1}{2}) + \alpha \xi^2 + \beta \xi^{\dagger 2}, \quad \alpha \neq \beta. \quad (14)$$

where parameters $\omega, \alpha, \beta \in \mathbb{R}$ and it was shown that the eigenvalues are real and positive $\omega^2 - 4\alpha\beta > 0$ when ξ represents linear harmonic oscillator annihilation operator. Now, (14) can be cast in a differential operator form:

$$H_- = -\bar{\omega}\frac{d}{dx}a^2(x)\frac{d}{dx} + b_1(x)\frac{d}{dx} + c_1(x) \quad (15)$$

where $\bar{\omega} = \omega - \alpha - \beta$, $\bar{\omega} > 0$. One gives $b_1(x)$ and $c_2(x)$ as [28]

$$b_1(x) = (\alpha - \beta)a(x)(2b(x) - a'(x)) \quad (16)$$

and

$$\begin{aligned} c_1(x) = & (\omega + \alpha + \beta)b^2(x) - (\omega + 2\beta)a'(x)b(x) - (\omega - \alpha + \beta)a(x)b'(x) \\ & + \beta(a(x)a''(x) + a'^2(x)) - \delta a'(x) + \frac{\omega}{2}. \end{aligned} \quad (17)$$

Hermitian equivalent of (14) which is h_-

$$h_- = \rho H_- \rho^{-1} \quad (18)$$

obtained by using $\rho = \exp\left(-\frac{1}{2\bar{\omega}} \int dx \frac{b_1(x)}{a^2(x)}\right)$, then h_- takes the form [19, 28]

$$h_- = -\bar{\omega} \frac{d}{dx} a^2(x) \frac{d}{dx} + V_-(x) \quad (19)$$

where [28]

$$\begin{aligned} V_-(x) = & \left(\frac{(\alpha - \beta)^2}{\bar{\omega}} + \bar{\omega} + 2(\alpha + \beta) \right) b(x)(b(x) - a'(x)) - (\bar{\omega} + \alpha + \beta)a(x)b'(x) \\ & + \frac{\alpha + \beta}{2}a(x)a''(x) + \frac{1}{4} \left(\frac{(\alpha - \beta)^2}{\bar{\omega}} + 2(\alpha + \beta) \right) a'^2(x) + \frac{\bar{\omega} + \alpha + \beta}{2}. \end{aligned} \quad (20)$$

In order to factorize h_- , it is written as a product of two operators

$$h_- = \mathcal{A}^\dagger \mathcal{A}. \quad (21)$$

Let us give an ansatz for the operators we use in above

$$\mathcal{A} = \sqrt{\bar{\omega}}a(x)\frac{d}{dx} + \tilde{b}(x), \quad \mathcal{A}^\dagger = -\sqrt{\bar{\omega}}a(x)\frac{d}{dx} + \tilde{b}(x) - \sqrt{\bar{\omega}}a'(x). \quad (22)$$

where $\tilde{b}(x)$ and $a(x)$ are real and they can be chosen to generate $V_\pm(x)$. Thus, $V_-(x)$ becomes

$$V_-(x) = \tilde{b}^2(x) - \sqrt{\bar{\omega}} \frac{d}{dx} (a(x)\tilde{b}(x)). \quad (23)$$

As is well known, h_- is isospectral to its partner Hamiltonian h_+ that is

$$h_+ = \mathcal{A}\mathcal{A}^\dagger \quad (24)$$

and it can be expressed as

$$h_+ = -\bar{\omega} \frac{d}{dx} a^2(x) \frac{d}{dx} + V_+(x). \quad (25)$$

Now, the partner potential of $V_-(x)$ is

$$V_+(x) = \tilde{b}^2(x) + \sqrt{\bar{\omega}} \left(-\sqrt{\bar{\omega}}a(x)a(x)'' + a(x)\tilde{b}'(x) - \tilde{b}(x)a'(x) \right). \quad (26)$$

Exploiting the fact that the non Hermitian partner Hamiltonians H_{\pm} and their Hermitian counterparts h_{\pm} are related by a similarity transformation

$$H_{\pm} = \rho^{-1} h_{\pm} \rho, \quad (27)$$

then we can get H_+ as

$$H_+ = -\bar{\omega} \frac{d}{dx} a^2(x) \frac{d}{dx} + b_1(x) \frac{d}{dx} + \frac{b'_1}{2} - \frac{b_1^2}{4\bar{\omega}a^2} + V_+(x). \quad (28)$$

Finally, we can give a diagrammatic representation for a chain of Hamiltonians we used as follows

$$H_- \rightarrow h_- \rightarrow h_+ \rightarrow H_+ \quad (29)$$

such that we can arrive at partner Hamiltonian H_+ using this chain.

2.1 Factorization of the Hermitian counterparts of Swanson Hamiltonian

Some choices of $a(x)$ and $b(x)$ may lead to an effective potential model. In [23, 28], it was shown that when the commutator is constant, i.e $[\xi, \xi^\dagger] = 1$, then the relation between $a(x)$ and $b(x)$ can be given as [28]

$$b(x) = \frac{a'(x)}{2} + \int \frac{dx}{2a(x)}. \quad (30)$$

The more general case corresponds to $[\xi, \xi^\dagger] \neq 1$. Then, let us use the ansatz for $a(x)$ and $b(x)$

$$a(x) = x^2 \quad (31)$$

$$b(x) = \frac{1}{x} + \frac{cx}{x^2 + d} \quad (32)$$

in (20), then $V_-(x)$ becomes

$$V_-(x) = \frac{a_1}{x^2} + 2(a_3 + 2a_4)x^2 + (-2a_1 + a_2)(c + 1) + a_5 + c \frac{(2a_1 + a_1c + 2a_1d - 3a_2d)x^2 + 2a_1(1 + d) - a_2d}{(x^2 + d)^2} \quad (33)$$

where c, d are some real constants. Here a_i , $i = 1, 2, \dots, 5$ can be recalled from (20), i.e:

$$a_1 = \frac{(\alpha - \beta)^2}{\bar{\omega}} + \bar{\omega} + 2(\alpha + \beta), \quad a_2 = \bar{\omega} + \alpha + \beta, \quad a_3 = \frac{\alpha + \beta}{2} \quad (34)$$

$$a_4 = \frac{1}{4} \left(\frac{(\alpha - \beta)^2}{\bar{\omega}} + 2(\alpha + \beta) \right), \quad a_5 = \frac{\bar{\omega} + \alpha + \beta}{2}. \quad (35)$$

With an eye on what is to follow, let us first match (33) with (23) to factorize h_- , so another ansatz is introduced for $\tilde{b}(x)$ which is,

$$\tilde{b}(x) = \frac{\mu}{x} - \varrho x + \frac{\lambda x}{x^2 + d}. \quad (36)$$

At that case, $V_-(x)$ is written as

$$V_-(x) = \frac{\mu^2}{x^2} + \varrho(\varrho + 3\sqrt{\bar{\omega}})x^2 - \mu(2\varrho + \sqrt{\bar{\omega}}) + \lambda \frac{-(2\varrho + \sqrt{\bar{\omega}})x^4 + (-3d\sqrt{\bar{\omega}} + \lambda + 2\mu - 2d\varrho)x^2 + 2d\mu}{(x^2 + d)^2}. \quad (37)$$

Hence, we can give the coupling constants of $V_-(x)$ in (37) in terms of ω, α, β so that we can compare (37) and (33). At the same time, its partner $V_+(x)$ becomes

$$V_+(x) = \frac{\mu^2}{x^2} + (\varrho^2 + \varrho\sqrt{\bar{\omega}} - 2\bar{\omega})x^2 - \mu(2\varrho + 3\sqrt{\bar{\omega}}) + \lambda \frac{-(2\varrho + 3\sqrt{\bar{\omega}})x^4 + (\lambda + 2\mu - 2d\varrho - d\sqrt{\bar{\omega}} + 2d\mu)x^2 + 2d\mu}{(d + x^2)^2}. \quad (38)$$

After some straightforward algebra, partner potentials happen to be

$$V_-(x) = \frac{\mu^2}{x^2} + \varrho(\varrho + 3\sqrt{\bar{\omega}})x^2 - (\mu + \lambda)(\sqrt{\bar{\omega}} + 2\varrho) + \lambda \frac{(2\varrho d + 2\mu + \lambda - d\sqrt{\bar{\omega}})x^2 + d(2\mu + d\sqrt{\bar{\omega}} + 2\varrho d)}{(x^2 + d)^2} \quad (39)$$

$$V_+(x) = \frac{\mu^2}{x^2} + \varrho(\varrho + \sqrt{\bar{\omega}} - 2\bar{\omega})x^2 - (\mu + \lambda)(3\sqrt{\bar{\omega}} + 2\varrho) + \lambda \frac{(2\varrho d + 2\mu + \lambda + 5d\sqrt{\bar{\omega}})x^2 + d(2\mu + 3d\sqrt{\bar{\omega}} + 2\varrho d)}{(x^2 + d)^2}. \quad (40)$$

We seek for a solvable $V_+(x)$ which implies some parameter restrictions such that the non-polynomial part in (38) vanishes within these conditions below

$$\lambda = -2d\sqrt{\bar{\omega}}, \quad \mu = -\frac{d}{2}(2\varrho + 3\sqrt{\bar{\omega}}). \quad (41)$$

Herefrom, if we plug (41) into $V_+(x)$, it turns into

$$V_+ = \frac{\mu^2}{x^2} + (\varrho^2 + \varrho\sqrt{\bar{\omega}} - 2\bar{\omega})x^2 + 2d \left(\varrho + \frac{7\sqrt{\bar{\omega}}}{2} \right) \left(\varrho + \frac{3\sqrt{\bar{\omega}}}{2} \right). \quad (42)$$

The same procedure is followed for $V_-(x)$, it follows that

$$V_- = \frac{\mu^2}{x^2} + \varrho(\varrho + 3\sqrt{\bar{\omega}})x^2 + 2d \left(\varrho + \frac{7\sqrt{\bar{\omega}}}{2} \right) \left(\varrho + \frac{\sqrt{\bar{\omega}}}{2} \right) + 4\bar{\omega}d^2 \frac{3x^2 + d}{(x^2 + d)^2}. \quad (43)$$

Comparing (33) and (43), as we stated before, it may give us a chance to write μ, λ, ϱ in terms of ω, α, β . Then,

$$a_1 = \mu^2 = \frac{d^2}{4}(2\varrho + 3\sqrt{\bar{\omega}})^2 \quad (44)$$

$$2(a_3 + 2a_4) = \varrho(\varrho + 3\sqrt{\bar{\omega}}) \quad (45)$$

$$(a_2 - 2a_1)(c + 1) + a_5 = 2d(\varrho + \frac{7\sqrt{\bar{\omega}}}{2})(\varrho + \frac{\sqrt{\bar{\omega}}}{2}) \quad (46)$$

and

$$c(-3a_2d + 2a_1 + a_1c + 2a_1d) = 12\bar{\omega}d^2 \quad (47)$$

$$2a_1(1+d) - a_2d = 4\bar{\omega}d^3. \quad (48)$$

We can solve c using last two relations above:

$$c_{1,2} = \frac{1}{2a_1} \left(2a_2d - 4\bar{\omega}d^3 \pm \sqrt{(4\bar{\omega}d^3 - 2a_2d)^2 + 48a_1\bar{\omega}d^2} \right). \quad (49)$$

Using (44), (45) and (46), ϱ can be found as

$$\varrho = \frac{1}{2}(\pm\sqrt{4X - 27\bar{\omega}} - 4\sqrt{\bar{\omega}}) \quad (50)$$

where

$$X = \frac{4a_1}{d^2} - 8(a_3 + 2a_4) + \frac{1}{2d}((a_2 - 2a_1)(c + 1) + a_5). \quad (51)$$

There are some restrictions on parameters such as $\varrho > 0$, so we take $\varrho = \frac{1}{2}(\sqrt{4X - 27\bar{\omega}} - 4\sqrt{\bar{\omega}})$. Inside the square root in (50) must be positive, ϱ is positive, then $4X > 43\bar{\omega}$. Because $a_2 - 2a_1 = -\frac{\bar{\omega}^2 + 3(\alpha + \beta)\bar{\omega} + 2(\alpha - \beta)^2}{\bar{\omega}} < 0$, then we need $c < 0, |c| > 1$ and we use the negative sign for c which is $c = \frac{1}{2a_1}(2a_2d - 4\bar{\omega}d^3 - \sqrt{(4\bar{\omega}d^3 - 2a_2d)^2 + 48a_1\bar{\omega}d^2})$. When (50), (49) are satisfied, then Hermitian counterpart of H_- which is h_- given by (19) is factorized.

2.2 Pseudosupersymmetry

Here, H_- is diagonalizable with a discrete spectrum which admits a set of complete biorthonormal eigenvectors $\{|\Psi_n\rangle, |\Phi_n\rangle\}$:

$$H_-|\Psi_n\rangle = E_n|\Psi_n\rangle, \quad H_-^\dagger|\Phi_n\rangle = E_n^*|\Phi_n\rangle \quad (52)$$

$$\sum_n |\Phi_n\rangle\langle\Psi_n| = \sum_n |\Psi_n\rangle\langle\Phi_n| = 1. \quad (53)$$

On the other hand, the intertwining relations between h_\pm can be written as

$$h_- \mathcal{A}^\dagger = \mathcal{A}^\dagger h_+ \quad (54)$$

and

$$h_+ \mathcal{A} = \mathcal{A} h_- . \quad (55)$$

And, H_- given in (14) is a non-Hermitian and diagonalizable Hamiltonian with real or complex conjugate eigenvalues. If there exists an operator η_1 such that

$$\eta_1 H_- = H_+ \eta_1 \quad (56)$$

where H_+ is the partner Hamiltonian of H_- , then, the intertwining operator η_1 is given by $\eta_1 = \rho^{-1} \mathcal{A} \rho$ where \mathcal{A} and its adjoint satisfy (54) and (55). This relation $\eta_1 = \rho^{-1} \mathcal{A} \rho$ was proven before [31]. The

intertwining operator that links a non-Hermitian Hamiltonian to the adjoint of its pseudo-supersymmetric partner Hamiltonian was also studied [32]. If one uses the similarity transformation

$$H_+ = \rho^{-1} h_+ \rho \quad (57)$$

then, if we multiply (57) by $\rho^{-1} \mathcal{A}$ from right, we have:

$$H_+ \rho^{-1} \mathcal{A} = \rho^{-1} h_+ \mathcal{A} \quad (58)$$

and if we use (55) in (58), we get

$$\rho^{-1} \mathcal{A} h_- = H_+ \rho^{-1} \mathcal{A} \quad (59)$$

$$\rho^{-1} \mathcal{A} \rho \rho^{-1} h_- \rho = H_+ \rho^{-1} \mathcal{A} \rho \Rightarrow \quad (60)$$

$$\rho^{-1} \mathcal{A} \rho H_- = H_+ \rho^{-1} \mathcal{A} \rho. \quad (61)$$

Then, it is seen that the intertwining operator η_1 is given by $\eta_1 = \rho^{-1} \mathcal{A} \rho$. Following the same way we can obtain $\eta_2 H_+ = H_- \eta_2$ where $\eta_2 = \rho^{-1} \mathcal{A}^\dagger \rho$. Here, η_1 and its pseudo-adjoint η_2 can be shown as

$$\eta_2^\dagger = \eta^{-1} \eta_2^\dagger \eta = \rho^{-1} \mathcal{A} \rho = \eta_1 \quad (62)$$

which leads to construct pseudo- super algebra of non-Hermitian supersymmetry. Thus, the operators Q, Q^\dagger become

$$Q = \begin{pmatrix} 0 & \eta_2 \\ 0 & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & 0 \\ \eta_1 & 0 \end{pmatrix}. \quad (63)$$

The pseudo- super Hamiltonian can be introduced as

$$\mathcal{H} = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} \quad (64)$$

where pseudo- super charges satisfy

$$\mathcal{H} = [Q, Q^\dagger]_+ = 0. \quad (65)$$

Now, let us find η_1 for our problem, it can be written as

$$\eta_1 = \sqrt{\bar{\omega}} x^2 \frac{d}{dx} + \frac{1}{\sqrt{\bar{\omega}} x (d + x^2)} ((\alpha - \beta - \varrho \sqrt{\bar{\omega}}) x^4 + ((\alpha - \beta)(d - c - 1) - (\frac{7d\bar{\omega}}{2} + 2\varrho d \sqrt{\bar{\omega}})) x^2 + d(\mu \sqrt{\bar{\omega}} - \alpha + \beta)) \quad (66)$$

which connects two partner Hamiltonians H_\pm . We can arrive at H_+ using η_1 given above.

3 Solutions

In this section, we aim to obtain the solutions of the partner Hamiltonians with (42) and (43). Let eigenfunctions of h_\pm and \tilde{h}_\pm be $\psi^\pm(x)$ and $\tilde{\varphi}^\pm(x)$. Then, h_\pm can be transformed into \tilde{h}_\pm ,

$$\tilde{h}_\pm = \theta h_\pm \theta^{-1} = -\bar{\omega} a^2(x) \frac{d^2}{dx^2} - \bar{\omega} a(x) a'(x) \frac{d}{dx} + V_\pm(x) + \frac{\bar{\omega}}{4} a'^2(x) + \frac{\bar{\omega}}{2} a(x) a''(x) \quad (67)$$

where $\theta = \sqrt{a(x)}$. One may use the change of independent variable in (67) as,

$$z = \int^x \frac{dy}{\sqrt{\bar{\omega}a(y)}} \quad (68)$$

and for our problem it equals to $z = -\frac{1}{\sqrt{\bar{\omega}x}}$. Then we arrive at

$$\tilde{h}_{\pm} = -\frac{d^2}{dz^2} + \left(V^{\pm}(x) + \frac{\bar{\omega}}{4}a'^2(x) + \frac{\bar{\omega}}{2}a(x)a''(x) \right)_{x \rightarrow z} \quad (69)$$

where we can write partner potentials $\tilde{V}_{\pm} = V_{\pm} + \frac{\bar{\omega}}{4}a'^2(x) + \frac{\bar{\omega}}{2}a(x)a''(x)$ in terms of z as

$$\tilde{V}_+(z) = \mu^2 \bar{\omega} z^2 + \frac{\varrho^2 + \sqrt{\bar{\omega}}\varrho}{\bar{\omega}} \frac{1}{z^2} + 2d(\varrho + \frac{7\sqrt{\bar{\omega}}}{2})(\varrho + \frac{3\sqrt{\bar{\omega}}}{2}) \quad (70)$$

and

$$\tilde{V}_-(z) = \mu^2 \bar{\omega} z^2 + \frac{\varrho^2 + 3\sqrt{\bar{\omega}}\varrho + 2\bar{\omega}}{\bar{\omega}} \frac{1}{z^2} + 2d(\varrho + \frac{7\sqrt{\bar{\omega}}}{2})(\varrho + \frac{\sqrt{\bar{\omega}}}{2}) + 4\bar{\omega}d + 4\bar{\omega}d \frac{2d\bar{\omega}z^2 - 1}{(d\bar{\omega}z^2 + 1)^2}. \quad (71)$$

We know that (70) and (71) are isospectral. In [36], an eigenvalue equation which is given by

$$\left(-\frac{d^2}{dz^2} + \frac{A}{z^2} + Bz^2 \right) \chi_n(z) = \epsilon_n \chi_n(z) \quad (72)$$

has exact eigenvalues and eigenfunctions as

$$\epsilon_n = 2\delta(2n + \gamma), \quad n = 0, 1, 2, \dots \quad (73)$$

where $\gamma = 1 + \frac{1}{2}\sqrt{1+4A}$, $\delta = \sqrt{B}$ and

$$\chi_n(z) = N z^{\gamma-1/2} e^{-\delta z^2/2} {}_1F_1(-n; \gamma; \delta z^2), \quad n = 0, 1, 2, \dots \quad (74)$$

This model was studied by Goldman and Krivchenkov[37] who showed that the energy spectrum of this potential is an infinite set of equidistant energy levels. Then, if we use $\hat{\omega} = \frac{d\sqrt{\bar{\omega}}}{2}(2\varrho + 3\sqrt{\bar{\omega}})$ and $\gamma = \frac{\varrho + \sqrt{\bar{\omega}}}{\sqrt{\bar{\omega}}} + \frac{1}{2}$ for the potential $\tilde{V}_+(z)$, exact eigenvalues and eigenfunctions of $\tilde{V}_+(z)$ are given by

$$E_n^+ = 2\hat{\omega}(2n + \frac{2\rho}{\sqrt{\bar{\omega}}} + 5) \quad (75)$$

and

$$\tilde{\varphi}_n^+(z) = C_n z^{\gamma-1/2} e^{-\frac{1}{2}\hat{\omega}z^2} {}_1F_1(-n; \gamma; \hat{\omega}z^2) \quad (76)$$

where C_n is the normalization constant which is given by [36]

$$C_n = (-1)^n \sqrt{\frac{2\hat{\omega}^\gamma(\gamma)_n}{n!\Gamma(\gamma)}} \quad (77)$$

and the solutions $\tilde{\varphi}_n^+(z)$ are square integrable on $(0, \infty)$ [36]. The relation between the confluent hypergeometric function and Laguerre polynomials may be used later:

$${}_1F_1(-n; b+1; y) = \frac{n!}{(b+1)_n} L_n^b(y). \quad (78)$$

Here the Pochhammer symbol $(s)_n = \frac{\Gamma(s+n)}{\Gamma(s)}$ is used for both (77) and (78). Now, in order to find the eigenfunctions of \tilde{h}_- we can follow an algebraic way such that we can start with a mapping between h_\pm and \tilde{h}_\pm introduced before:

$$\tilde{h}_\pm = \theta h_\pm \theta^{-1}. \quad (79)$$

Also, this relation can be written as

$$\begin{aligned} \tilde{h}_+ &= \theta \mathcal{A} \mathcal{A}^\dagger \theta^{-1} \\ &= (\theta \mathcal{A} \theta^{-1}) (\theta \mathcal{A}^\dagger \theta^{-1}) \\ &= \tilde{\mathcal{A}} \tilde{\mathcal{A}}^\dagger \end{aligned} \quad (80)$$

and following the same way one obtains

$$\tilde{h}_- = \tilde{\mathcal{A}}^\dagger \tilde{\mathcal{A}} \quad (81)$$

where $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}^\dagger$ can be introduced as

$$\tilde{\mathcal{A}} = \sqrt{\bar{\omega}} a(x) \frac{d}{dx} + \tilde{b}(x) - \frac{\sqrt{\bar{\omega}}}{2} a'(x) \quad (82)$$

$$\tilde{\mathcal{A}}^\dagger = -\sqrt{\bar{\omega}} a(x) \frac{d}{dx} + \tilde{b}(x) - \frac{\sqrt{\bar{\omega}}}{2} a'(x). \quad (83)$$

Finally we can use $\tilde{\mathcal{A}}^\dagger$ to find $\varphi_n^-(z)$ which is the solution of $\tilde{V}_-(z)$:

$$\tilde{\varphi}_n^-(z) = \tilde{\mathcal{A}}^\dagger \tilde{\varphi}_n^+(z) = \left(-\sqrt{\bar{\omega}} a(x) \frac{d}{dx} + \tilde{b}(x) - \frac{\sqrt{\bar{\omega}}}{2} a'(x) \right)_{x \rightarrow z} C'_n z^{\gamma-1/2} e^{-\frac{1}{2}\hat{\omega} z^2} L_n^{\gamma-1}(\hat{\omega} z^2) \quad (84)$$

$$= \left(-\frac{d}{dz} + \hat{\omega} z + \frac{\varrho}{\sqrt{\bar{\omega}}} \frac{1}{z} + \frac{2d\bar{\omega}z}{1+d\bar{\omega}z^2} \right) C'_n z^{\gamma-1/2} e^{-\frac{1}{2}\hat{\omega} z^2} L_n^{\gamma-1}(\hat{\omega} z^2) \quad (85)$$

where we use μ and λ from (41) in (84). Using a relation [10]

$$\frac{d}{dt} L_n^\beta(t) = -L_{n-1}^{\beta+1}(t) \quad (86)$$

and the identity of Laguerre polynomials

$$L_n^\beta(t) = L_{n-1}^\beta + L_n^{\beta-1}(t) \quad (87)$$

helps us to re-write $\varphi_n^-(z)$ as [10]

$$\tilde{\varphi}_n^-(z) = \frac{2C'_n z^{\gamma+1/2} e^{-\hat{\omega} z^2/2}}{z^2 + (d\bar{\omega})^{-1}} [(\gamma + n + 1) L_n^{\gamma-1}(\hat{\omega} z^2) - (n + 1) L_{n+1}^{\gamma-1}(\hat{\omega} z^2) + \gamma L_n^\gamma(\hat{\omega} z^2)]. \quad (88)$$

On the other hand, one can show that the solutions $\psi_n^\pm(z)$ can be normalized. For example, ψ_n^+ can be given as

$$\psi_n^+(z) = \mathcal{N} z^{\gamma+1/2} e^{-\frac{1}{2}\hat{\omega}z^2} {}_1F_1(-n; \gamma; \hat{\omega}z^2) \quad (89)$$

where \mathcal{N} is the normalization constant. To normalize ψ_n^+ , we may introduce the integral [38]

$$\int_0^\infty u^{2v-1} e^{-ju^2} {}_1F_1(-n; v; ju^2) {}_1F_1(-m; v; ju^2) du = \frac{1}{2} \frac{n! \Gamma(v)}{j^v (v)_n} \delta_{mn} \quad (90)$$

for $v > 0$, $m, n = 0, 1, 2, \dots$ and $\delta_{mn} = 0$ if $m \neq n$, $\delta_{mn} = 1$ if $m = n$. Now we use

$${}_1F_1(-n; b; y) = \sum_{k=0}^n \frac{(-n)_k y^k}{(b)_k k!} \quad (91)$$

then we have

$$J_{mn} = \sum_{k=0}^n \sum_{l=0}^m \frac{(-m)_l (-n)_k \hat{\omega}^{k+l}}{(\gamma)_k (\gamma)_l k! l!} \int_0^\infty z^{2\gamma+2k+2l+1} e^{-\hat{\omega}z^2} dz. \quad (92)$$

The integral representation of the gamma function is

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x > 0, \quad (93)$$

and using a variable change, we obtain

$$J_{mn} = \frac{1}{2\hat{\omega}^{\gamma+1}} \sum_{k=0}^n \left(\sum_{l=0}^m \frac{(-m)_l (\gamma+k+1)_l}{(\gamma)_l l!} \right) \frac{(-n)_k \Gamma(\gamma+k+1)}{(\gamma)_k k!}. \quad (94)$$

The term in the bracket in above relation corresponds to hypergeometric function ${}_2F_1$, then we re-write J_{mn} as

$$J_{mn} = \frac{1}{2\hat{\omega}^{\gamma+1}} \sum_{k=0}^n {}_2F_1(-m; \gamma+k+1; \gamma; 1) \frac{(-n)_k \Gamma(\gamma+k+1)}{(\gamma)_k k!}. \quad (95)$$

Using Chu-Vandermonde identity which is given by [36]

$${}_2F_1(a; b; c; 1) = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \quad (96)$$

in (95), we get

$$J_{mn} = \frac{\Gamma(\gamma)}{2\hat{\omega}^{\gamma+1}} \sum_{k=0}^n \frac{(-n)_k \Gamma(k+\gamma+1)}{(\gamma)_k k!} \frac{\Gamma(m-k-1)}{\Gamma(\gamma+m) \Gamma(-k-1)}. \quad (97)$$

Using the identity [36]

$$(-k)_n = \begin{cases} \frac{(-1)^n k!}{(k-n)!}, & 0 \leq n \leq k; \\ 0, & n > k. \end{cases} \quad (98)$$

and (97), we obtain

$$J_{mn} = \frac{(n+\gamma)(n+1)! \Gamma(\gamma)}{2\hat{\omega}^{\gamma+1} (\gamma)_n} \quad (99)$$

for $k = n = m$. Finally, normalization constant becomes

$$\mathcal{N} = (-1)^n \sqrt{\frac{2\hat{\omega}^{\gamma+1} (\gamma)_n}{\hat{\omega} (n+\gamma)(n+1)! \Gamma(\gamma)}}. \quad (100)$$

4 Conclusion

To conclude, we have derived a new class of isotonic nonlinear oscillator using the concepts of pseudo-supersymmetry. We have studied a generalized non-Hermitian Hamiltonian and its partner that can generate solvable isotonic and nonlinear oscillator potentials by using an appropriate mapping, i.e. a similarity transformation. Hermitian partner Hamiltonians which include effective solvable new isotonic and its nonlinear partner potential are factorized and it is seen that the nonlinear oscillator is solvable under parameter restrictions given by (41). We have also checked that the intertwining operator η_1 that connects non-Hermitian partner Hamiltonians H_{\pm} can be given in terms of a transformation of the factorization operator \mathcal{A} as $\eta_1 = \rho^{-1}\mathcal{A}\rho$. At the end, we give solutions of isotonic nonlinear oscillator after a coordinate and eigenfunction transformations. We have seen that the eigenvalues are real and positive which agrees with the results of [22]. We introduce formulae for all the discrete eigenvalues and normalized eigenfunctions.

References

- [1] A de Souza Dutra, Phys. Rev. A 47 (1993) R2435; A de Souza Dutra and H. Boschi-Filho, Phys. Rev. A 50 (1994) 2915.
- [2] C. Grosche, J. Phys. A: Math. and Gen. 28 (1995) 5889; 29 (1996) 365.
- [3] G. Junker and P. Roy, Phys. Lett. A 232 (1997) 155.
- [4] F. Calogero, J. Math. Phys. 10 (1969) 2191.
- [5] Z. Dongpei, J. Phys. A: Math. Gen. 20 (1987) 4331.
- [6] J. F. Cariñena, A. M. Perelemov, M. F. Rañada and M. Santander, J. Phys. A: Math.Theor. 41 (2008) 085301.
- [7] J. M. Fellows and R. A. Smith, J. Phys. A: Math.Theor. 42 (2009) 335303.
- [8] J. Sesma, J. Phys. A: Math.Theor. 43 (2010) 185303.
- [9] R. A. Kraenkel and M. Senthilvan J. Phys. A: Math.Theor. 42 (2009) 415303.
- [10] R. L. Hall, N. Saad, Ö. Yeşiltaş, J. of Phys. A: Math. and Theo. 43 (2010) 465304.
- [11] Y. Grandati, Solvable rational extensions of the isotonic oscillator, Ann. of Phys., in press; Y. Grandati and A. Bérard, Ann. of Phys. 325 (2010) 1235.
- [12] O. A. Chalykh, A. P. Veselov, J. Nonlinear Math. Phys. 12 179.
- [13] Y. Weissman, J. Jortner, Phys. Lett. A 70 (1979) 177.
- [14] A. Mostafazadeh, J. Math. Phys. 43 (2002) 205; A. Mostafazadeh, J. Math. Phys. 43 (2002) 2814;
- [15] A. Mostafazadeh and G. Sclarici, Phys. Lett. A 374 (2010) 2401.
- [16] B. E. Mossallam and M. R. Ejtehadi, J. Chem. Phys. 128 (2008) 125106.

- [17] L. Jin and Z. Song, Phys. Rev. A 80 (2009) 052107.
- [18] A. Mostafazadeh, Nucl. Phys. B 640 (2002) 419.
- [19] B. Bagchi, C. Quesne and R. Roychoudhury, J. Phys. A: Math. Gen. 38 (2005) L647.
- [20] R. Roychoudhury and B. Roy, Phys. Lett. A 361 (2007) 291.
- [21] Ö. Yeşiltaş, J. Phys. A: Math. Theor. 43 (2010) 095305.
- [22] M. S. Swanson, J. Math. Phys. 45 (2004) 585.
- [23] C. Quesne, J. of Phys. A 41 (2008) 244022.
- [24] Paulo E. G. Assis, A. Fring, J. Phys. A: Math. Theor. 42 (2009) 015203.
- [25] Paulo E. G. Assis, arXiv:1012.0194.
- [26] O. Cherbal, M. Drir, M. Maamache, D. A. Trifonov, SIGMA 6 (2010) 096.
- [27] D. P. Musumbu, H. B. Geyer and W. D. Heiss, J. Phys. A. 40 F75 (2007).
- [28] B. Bagchi and T. Tanaka, Phys. Lett. A 372 (2008) 5390.
- [29] A. Sinha and P. Roy, J. Phys. A: Math. Theo. 41 (2008) 335306.
- [30] A. Mostafazadeh, Nucl. Phys. B 640 (2002) 419.
- [31] D Bazeia, A Das, L Greenwood et al, Phys. Lett. B 673 (2009) 283.
- [32] R. Roychoudhury, B. Roy, Phys. Lett. A 361 (2007) 291.
- [33] F. Cooper, A. Khare and U. Sukhatme, Supersymmetry in Quantum Mechanics, Singapore: World Scientific, 2001.
- [34] A. R. Plastino et al, Phys. Rev. A 60 (1999) 4318.
- [35] L. Solombrino, J. Math. Phys. 43 (2002) 5439.
- [36] R. L. Hall, N. Saad and A. B. von Keviczky, J. Math. Phys. 43 (2002) 94.
- [37] I. I. Goldman, V. D. Krivchenkov, Problems in Quantum Mechanics, Pergamon Press: London, 1961; Chapter 1 (Problem 11).
- [38] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables, DoverPubl. Inc., New York, 1972.